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On a Certain Class of Linear Differential Equations.

By Thomas Craig.

The subject of linear differential equations whose coefficients are singly or doubly periodic functions of the first kind has been studied by Picard* and Floquet,† and it has been shown that such equations always admit of at least one integral which is a periodic function of the second kind.

The case where the coefficients are periodic functions of the second kind does not seem to have been attempted by any one—and indeed in the general case the problem seems almost impossible of solution—though when the multipliers are roots of unity the solution (at least for singly periodic functions) can be easily led back to the case where the coefficients are periodic of the first kind.

In the following paper I have determined the conditions which are necessary in order that a linear differential equation shall admit of an integral which is a periodic function of the third kind—in the beginning I limit myself to singly periodic functions.

I shall start with the case of the linear differential equation of the second order,

$$1. \qquad \frac{d^2y}{dx^2} + p_1 \frac{dy}{dx} + p_2 y = 0,$$

where the coefficients p_1 and p_2 are uniform functions of x. What must be the form of these functions in order that (1) may have an integral which is a periodic function of the third kind?

Suppose $y = F(x)$ to be such an integral, and $F(x + \omega) = e^{-\lambda x + \lambda_0}$ where λ and λ_0 are constants; then if we change x into $x + \omega$ we must have that $F(x + \omega)$ is an integral of

$$2. \qquad \frac{d^2y}{dx^2} + p_1(x + \omega) \frac{dy}{dx} + p_2(x + \omega) y = 0.$$

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* Crelle, Vol. XC. Sur les équations différentielles linéaires à coëfficients doublement périodiques.

† Annales de l'Ecole Normale, Feb. 1883. Sur les équations différentielles linéaires à coëfficients périodiques. *Ibid.* May, 1884. Sur les équations différentielles linéaires à coëfficients doublement périodiques.

Writing then in (2) $y = F(x+\omega)$, $= e^{-\lambda x + \lambda_0} F(x)$ we have, writing for brevity F instead of $F(x)$,

$$\frac{d^2F}{dx^2} + [-2\lambda + p_1(x+\omega)]\frac{dF}{dx} + [\lambda^2 - \lambda p_1(x+\omega) + p_2(x+\omega)]F = 0,$$

also
$$\frac{d^2F}{dx^2} + p_1(x)\frac{dF}{dx} + p_2 F = 0.$$

Subtracting we have

$$[-2\lambda + p_1(x+\omega) - p_1(x)]\frac{dF}{dx} + [\lambda^2 - \lambda p_1(x+\omega) + p_2(x+\omega) - p_2(x)]F = 0,$$

which must be satisfied identically; therefore

3.
$$\begin{aligned} p_1(x+\omega) &= p_1(x) + 2\lambda \\ p_2(x+\omega) &= p_2(x) + \lambda p_1(x) + \lambda^2 \end{aligned}$$

and obviously

4.
$$\begin{aligned} p_1(x+n\omega) &= p_1(x) + 2n\lambda \\ p_2(x+n\omega) &= p_2(x) + n\lambda p_1(x) + n^2\lambda^2. \end{aligned}$$

The first of equations (3) is obviously satisfied by $p_1(x) = \frac{2\lambda x}{\omega}$ and the second by $p_2(x) = \frac{\lambda^2 x^2}{\omega^2}$. If we use these values for p_1 and p_2 we have the equation (replacing λ by $-\mu$).
$$\frac{d^2y}{dx^2} - \frac{2\mu x}{\omega}\frac{dy}{dx} + \frac{\mu^2 x^2}{\omega^2}y = 0,$$

of which the general integral is

$$y = e^{\frac{\mu x^2}{2\omega}}\left[C_1 \cos\left(x\sqrt{\frac{\mu}{\omega}}\right) + C_2 \sin\left(x\sqrt{\frac{\mu}{\omega}}\right)\right]$$

and on giving μ the value $\frac{4\pi^2}{\omega}$, *i. e.*, $\lambda = -\frac{4\pi^2}{\omega}$ this is obviously a periodic function of the third kind. Changing x into $x+\omega$ and it is easy to verify that

$$y(x+\omega), = e^{\frac{\mu(x+\omega)^2}{2\omega}}\left\{C_1 \cos\left(x\sqrt{\frac{\mu}{\omega}} + \sqrt{\mu\omega}\right) + C_2 \sin\left(x\sqrt{\frac{\mu}{\omega}} + \sqrt{\mu\omega}\right)\right\}$$

satisfies the equation

$$\frac{d^2y}{dx^2} - \frac{2\mu}{\omega}(x+\omega)\frac{dy}{dx} + \frac{\mu^2}{\omega^2}(x+\omega)^2 y = 0.$$

The most general values of p_1 and p_2 are easily found. Suppose $\phi_j^{(i)}(x)$ to be a singly periodic function of the first kind, *i. e.* $\phi_j^{(i)}(x+\omega) = \phi_j^{(i)}(x)$ and give i the values $2, 3 \ldots n$ and j the values $1, 2 \ldots n$. It is obvious now that the most general values of p_1 and p_2 are given by

$$p_1(x) = \phi_1^{(2)}(x) + \frac{2\lambda x}{\omega},$$

$$p_2(x) = \phi_2^{(2)}(x) + \frac{\lambda\phi_1^{(2)}(x)}{\omega}x + \frac{\lambda^2 x^2}{\omega^2}.$$

Take as another illustration the equation of the third order

$$\frac{d^3y}{dx^3}+p_1\frac{d^2y}{dx^2}+p_2\frac{dy}{dx}+p_3y=0.$$

Here we must have

$$\begin{aligned}p_1(x+\omega)&=p_1(x)+3\lambda\\ p_2(x+\omega)&=p_2(x)+2\lambda p_1(x)+3\lambda^2\\ p_3(x+\omega)&=p_3(x)+\lambda p_2(x)+\lambda^2p_1(x)+\lambda^3,\end{aligned}$$

these are satisfied by the values

$$p_1(x)=\frac{3\lambda x}{\omega},\ p_2(x)=\frac{3\lambda^2x^2}{\omega^2},\ p_3(x)=\frac{\lambda^3x^3}{\omega^3}.$$

Replacing λ by $-\mu$ we have the equation

$$\frac{d^3y}{dx^3}-\frac{3\mu x}{\omega}\frac{d^2y}{dx^2}+\frac{3\mu^2x^2}{\omega^2}\frac{dy}{dx}-\frac{\mu^3x^3}{\omega^3}y=0,$$

the general integral of which is

$$y=e^{\frac{\mu x^2}{2\omega}}\left\{C_1\cos\left(x\sqrt{3\frac{\mu}{\omega}}\right)+C_2\sin\left(x\sqrt{\frac{3\mu}{\omega}}\right)+C_3\right\}$$

which for $\mu=\frac{4\pi^2}{3\omega}$ is a function of the third kind. The general integrals of these two particular equations of the second and of the third orders are thus periodic functions of the third kind—this does not hold in the case of an equation of this form of any higher order: for example take the equation

$$\frac{d^4y}{dx^4}-4\frac{\mu x}{\omega}\frac{d^3y}{dx^3}+\frac{6\mu^2x^2}{\omega^2}\frac{d^2y}{dx^2}-4\frac{\mu^3x^3}{\omega^3}\frac{dy}{dx}+\frac{\mu^4x^4}{\omega^4}y=0,$$

the general integral is

$$y=e^{\frac{\mu x^2}{2\omega}}[A\cos(m_1x+\alpha)+B\cos(m_2x+\beta)],$$

A, B, α, β being constants and m_1^2, m_2^2 the roots with their signs changed of the equation

$$\xi^2+\frac{6\mu}{\omega}\xi+\frac{3\mu^2}{\omega^2}=0.$$

The general integral in this case is not a periodic function of the third kind—but each of the particular integrals

$$e^{\frac{\mu x^2}{2\omega}}\cos(m_1x+\alpha),\quad e^{\frac{\mu x^2}{2}}\cos(m_2x+\alpha)$$

is such a function if λ be properly determined. The general form of the coefficients in any case is readily found. Take the equation of the n^{th} order

$$P(x)\equiv\frac{d^ny}{dx^n}+p_1\frac{d^{n-1}y}{dx^{n-1}}+p_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+p_ny=0.$$

If $F(x)$ be an integral of $P(x) = 0$, then $F(x+\omega)$ will be an integral of $P(x+\omega) = 0$.

Expressing the necessary conditions for this and writing $F(x+\omega) = e^{-\lambda x} F(x)$ we find

$$p_1(x + l\omega) = p_1(x) + nl\lambda$$

$$p_2(x + l\omega) = p_2(x) + (n-1)\, l\lambda p_1(x) + \frac{n(n-1)}{1.2}\, l^2\lambda^2$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\begin{aligned} p_i(x + l\omega) = {} & p_i(x) + (n-i+1)\, l\lambda p_{i-1}(x) + \frac{(n-i+2)(n-i+1)}{1.2}\, l^2\lambda^2 p_{i-2}(x) \\ & + \ldots \frac{(n-i+\alpha)(n-i+\alpha-1)\ldots(n-i+1)}{\alpha!}\, l^\alpha \lambda^\alpha p_{i-\alpha}(x) \\ & + \ldots \frac{n.n-1\ n-2\ldots n-i+1}{i!}\, l^i\lambda^i \end{aligned}$$

$$p_n(x + l\omega) = p_n(x) + l\lambda p_{n-1}(x) + l^2\lambda^2 p_{n-2}(x) + \ldots + l^n\lambda^n$$

where l is any integer; the general term may be written briefly

$$p_i(x + l\omega) = \sum_{\alpha=0}^{\alpha=i} \frac{n-i+\alpha\,|\,n-i+1}{\alpha!}\, l^\alpha\lambda^\alpha p_{i-\alpha}(x).$$

These conditions are all satisfied if we write

$$p_i(x) = \frac{n.n-1\ldots n-i+1}{i!}\, \frac{\lambda^i x^i}{\omega^i}$$

i. e.

$$p_1(x) = \frac{n\lambda x}{\omega},\quad p_2(x) = \frac{n.n-1}{1.2}\, \frac{\lambda^2 x^2}{\omega^2},\ \text{etc.}$$

the general values for the coefficients are now easily seen to be

$$p_1(x) = \phi_1^{(n)}(x) + \frac{n\lambda x}{\omega}$$

$$p_2(x) = \phi_2^{(n)}(x) + (n-1)\frac{\lambda\varphi_1^{(n)}(x)}{\omega}\, x + \frac{n.n-1}{1.2}\, \frac{\lambda^2 x^2}{\omega^2}$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\begin{aligned} p_i(x) = {} & \phi_i^{(n)}(x) + (n-i+1)\frac{\lambda\varphi_{i-1}^{(n)}(x)}{\omega}\, x + \frac{n-i+1.n-i+2}{1.2}\, \frac{\lambda^2\varphi_{i-2}^{(n)}(x)}{\omega^2}\, x^2 \\ & + \ldots + \frac{n-i+\alpha.n-i+\alpha-1\ldots n-i+1}{\alpha!}\, \frac{\lambda^\alpha\varphi_{i-\alpha}^{(n)}(x)}{\omega^\alpha}\, x^\alpha \\ & + \ldots + \frac{n.n-1\ldots n-i+1}{i!}\, \frac{\lambda^i x^i}{\omega^i} \end{aligned}$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$p_n(x) = \phi_n^{(n)}(x) + \frac{\lambda\varphi_{n-1}^{(n)}(x)}{\omega}\, x + \frac{\lambda^2\varphi_{n-2}^{(n)}(x)}{\omega^2}\, x^2 + \ldots + \frac{\lambda^n x^n}{\omega^n}$$

or briefly

$$p_i(x) = \sum_{\alpha=0}^{\alpha=i} \frac{n-i+\alpha|n-i+1}{\alpha\,!} \frac{\lambda^\alpha \varphi^{(n)}_{i-\alpha}(x)}{\omega^\alpha} x^\alpha$$

$$i = 1,\ 2,\ 3 \ldots n.$$

It remains now to show that a differential equation whose coefficients are of the kind just found possesses at least one integral which is a periodic function of the third kind. It will be quite sufficient to do this for equations of the second order. The equation

$$\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2 y = 0$$

has for coefficients

$$p_1(x) = P_1(x) + \frac{2\lambda x}{\omega}$$

$$p_2(x) = P_2(x) + \frac{\lambda P_1(x)}{\omega} x + \frac{\lambda^2 x^2}{\omega^2}$$

where $P_1(x)$ and $P_2(x)$ are periodic functions of the first kind having ω for their common period. Assume

$$y = e^{\frac{-\lambda x^2}{2\omega} + \lambda'} \phi(x),$$

where $\phi(x)$ is a simply periodic function of either the first or second kind. Substituting this value of y in the above differential equation gives

$$\frac{d^2\varphi}{dx^2} + \left(p_1 - \frac{2\lambda x}{\omega}\right)\frac{dy}{dx} + \left(p_2 - p_1\frac{\lambda x}{\omega} + \frac{\lambda^2 x^2}{\omega^2}\right)\phi = 0.$$

Introducing now the above values of p_1 and p_2 this becomes

$$\frac{d^2\varphi}{dx^2} + P_1\frac{d\varphi}{dx} + P_2\phi = 0.$$

Now Picard has shown that this equation always possesses at least one integral which is a periodic function of the second kind (becoming of the first kind for the multiplier $= 1$). We conclude therefore that it is always possible to find at least one integral of our given differential equation which is a periodic function of the third kind.

Next we seek to determine the maximum number of integrals of this kind that our equation may possess. In the differential equation

$$\frac{d^n y}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots + p_n y = 0.$$

Assume $y = e^{\frac{\lambda x^2}{2\omega}}\Phi$ where $\Phi(x)$ is, in general, a periodic function of the second kind, and give p_i the value

$$p_i = \sum_{\alpha=0}^{\alpha=i} \frac{n-i+\alpha|n-i+1}{\alpha\,!} \frac{\lambda^\alpha x^\alpha}{\omega^\alpha} \phi_{i-\alpha}(x)$$

where $\phi_{i-\alpha}$ is a periodic function of the first kind. The equation now becomes

$$\Phi \equiv \frac{d^n\Phi}{dx^n} + \phi_1 \frac{d^{n-1}\Phi}{dx^{n-1}} + \phi_2 \frac{d^{n-2}\Phi}{dx^{n-2}} + \ldots + \phi_n \Phi = 0.$$

The solution of our problem is therefore led to the investigation of a linear differential equation whose coefficients are singly periodic functions of the first kind. The theory of this equation has been completely developed by Floquet and it is only necessary therefore to quote his results.

Denote by $F_1(x)$, $F_2(x) \ldots F_n(x)$, n distinct solutions of $\Phi = 0$. If the variable describes any path from x to $x + \omega$ the uniform functions $F(x)$ become $F(x + \omega)$ while the coefficients ϕ_i resume their original values, consequently the functions $F_1(x + \omega)$, $F_2(x + \omega) \ldots F_n(x + \omega)$ constitute another fundamental system of integrals of $\Phi = 0$.

We have then

$$\begin{aligned} F_1(x+\omega) &= A_{11}F_1(x) + A_{12}F_2(x) + \ldots + A_{1n}F_n(x) \\ F_2(x+\omega) &= A_{21}F_1(x) + A_{22}F_2(x) + \ldots + A_{2n}F_n(x) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_n(x+\omega) &= A_{n1}F_1(x) + A_{n2}F_2(x) + \ldots + A_{nn}F_n(x) \end{aligned}$$

where the determinant

$$\begin{vmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ A_{n1} & A_{n2} & \ldots & A_{nn} \end{vmatrix}$$

does not vanish.

Suppose now that $\Phi = 0$ admits as integral a periodic function of the second kind, say $\mathbf{F}(x)$, where $\mathbf{F}(x + \omega) = \varepsilon \mathbf{F}(x)$, we have then

$$\mathbf{F}(x) = n_1 F_1(x) + n_2 F_2(x) + \ldots + n_n F_n(x),$$

where the constants n cannot all vanish. Now since $\mathbf{F}(x + \omega) = \mathbf{F}(x)$ we have immediately

$$F_1(x)\Sigma A_{i1}n_i + F_2(x)\Sigma A_{i2}n_i + \ldots + F_n(x)\Sigma A_{in}n_i = \varepsilon \Sigma n_i F_i(x).$$

This requires that the constants n satisfy the equations

$$\begin{aligned} (A_{11} - \varepsilon)n_1 + A_{21}n_2 + \ldots + A_{n1}n_n &= 0 \\ A_{12}n_1 + (A_{22} - \varepsilon)n_2 + \ldots + A_{n2}n_n &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ A_{1n}n_1 + A_{2n}n_2 + \ldots + (A_{nn} - \varepsilon)n_n &= 0, \end{aligned}$$

consequently

$$\Delta = \begin{vmatrix} A_{11} - \varepsilon & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} - \varepsilon & \dots & A_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ A_{1n} & A_{2n} & \dots & A_{nn} - \varepsilon \end{vmatrix} = 0 .$$

The equation $\Delta = 0$ is called the fundamental equation, and its roots, which are the multipliers of the periodic functions of the second kind, are evidently all different from zero. It is also well known that the roots of the fundamental equation are independent of the choice of the fundamental system of integrals, a fact which was proved by Fuchs.* Floquet has shown that the general integral of this equation is of the form

$$\begin{aligned} & \psi_{11}(x) + x\psi_{12}(x) + \dots + x^{\mu_1 - 1}\psi_{1\mu_1}(x) \\ + & \psi_{21}(x) + x\psi_{22}(x) + \dots + x^{\mu_2 - 1}\psi_{2\mu_2}(x) \\ + & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ + & \psi_{m1}(x) + x\psi_{m2}(x) + \dots + x^{\mu_m - 1}\psi_{2m\mu_m}(x), \end{aligned}$$

where ψ_{ik} is a periodic function of the second kind whose multiplier is ε_i. In this $\varepsilon_1 \varepsilon_2 \dots \varepsilon_m$ are the distinct roots of $\Delta = 0$ of the orders $\mu_1, \mu_2 \dots \mu_m$ respectively.

The general conclusions arrived at by Floquet concerning the equation $\Phi = 0$ are as follows:

I. Let $\varepsilon_1\ \varepsilon_2 \dots \varepsilon_m$ be the distinct roots of the fundamental equation $\Delta = 0$; let λ_i denote the order parting from which the minors of Δ cease to be *all* zero for $\varepsilon = \varepsilon_i$:

1^0 $\Phi = 0$ admits as distinct integrals

$$\lambda_1 + \lambda_2 + \dots \lambda_m$$

periodic functions of the second kind and no more;

2^0 There exists a fundamental system of solutions consisting of

$$\lambda_1 + \lambda_2 + \dots + \lambda_m$$

periodic functions of the second kind, and also

$$m - (\lambda_1 + \lambda_2 + \dots + \lambda_m)$$

expressions each of which is of the form of a polynomial in x with coefficients which periodic functions of the second kind all having the same multiplier;

3^0 The multipliers which appear in this fundamental system either as elements or as coefficients in the elements are equal to the different roots $\varepsilon_1, \varepsilon_2 \dots \varepsilon_m$ of the fundamental equation.

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[*] Crelle, Vol. LXVI, Zur Theorie der linearen Differentialgleichungen mit veränderlichen Coefficienten.

II. In order that $\Phi = 0$ may admit of n distinct integrals which are periodic functions of the second kind, it is necessary and sufficient that each root of $\Delta = 0$ shall annul all the minors of Δ up to the order which is equal to the degree of multiplicity of this root.

We have therefore in the case of the differential equation

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + p_n y = 0,$$

$$\left(\text{where } p_i = \sum_{\alpha=0}^{\alpha=i} \frac{n-i+\alpha \,|\, n-i+1}{\alpha\,!} \, \frac{\lambda^\alpha x}{\omega} \, \phi_{i-\alpha}(x)\right)$$

$$\lambda_1 + \lambda_2 + \ldots \lambda_m$$

linearly independent integrals which are periodic functions of the third kind. It remains now to find the remaining $n - (\lambda_1 + \lambda_2 + \ldots \lambda_m)$ integrals, which, with those just mentioned, constitute a fundamental system of the given equation.

It is easy, and, for present purposes, quite enough to verify what these integrals are. Assume

$$e^{\frac{-\lambda x^2}{2\omega}} y_i = e^{\frac{-\lambda x^2}{2\omega}} \left[\Theta_{i1}(x) + x\Theta_{i2}(x) + x^2\Theta_{i3}(x) + \ldots + x^{\mu_j - 1}\Theta_{i\mu_j}(x)\right],$$

give μ_j the meaning above assigned to it and let $\Theta_{ik}(x)$ denote a periodic function of the second kind having ε_i for a multiplier. Substitute this value in the differential equation

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + p_n y = 0,$$

and give $p_1, p_2 \ldots p_n$ the above values, viz.

$$p_i = \sum_{\alpha=0}^{\alpha=i} \frac{n-i+\alpha \,|\, n-i+1}{\alpha\,!} \, \frac{\lambda^\alpha x^\alpha}{\omega^\alpha} \, \phi_{i-\alpha}(x);$$

we have then $\quad \dfrac{d^n y_i}{dx^n} + \phi_1 \dfrac{d^{n-1} y_i}{dx^{n-1}} + \phi_2 \dfrac{d^{n-2} y_i}{dx^{n-2}} + \ldots + \phi_n y = 0,$

and this as we know possesses

$$n - (\lambda_1 + \lambda_2 + \ldots + \lambda_m)$$

integrals of the form

$$y_i = \Theta_{i1}(x) + x\Theta_{i2}(x) + x^2\Theta_{i3}(x) + \ldots + x^{\mu_j - 1}\Theta_{i\mu_j}(x).$$

The solution of the given differential equation is thus seen to depend in all cases upon the solution of an equation of the form

$$\frac{d^n y}{dx^n} + \phi_1 \frac{d^{n-1} y}{dx^{n-1}} + \phi_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots \phi_n y = 0,$$

where $\phi_1, \phi_2 \ldots \phi_n$ are singly periodic functions of the first kind. The results arrived at then are as follows:

I. In order that the linear differential equation

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + p_n y = 0,$$

may possess integrals which are periodic functions of the third kind, it is necessary and sufficient that the coefficients p be of the form

$$p_i = \sum_{\alpha=0}^{\alpha=i} \frac{n-i+\alpha \,|\, n-i+1}{\alpha\,!} \, \frac{\lambda^\alpha x^\alpha}{\omega^\alpha} \, \phi_{i-\alpha}(x),$$

i. e. of the form of polynomials in x whose coefficients, apart from certain determinate constants, are periodic functions of the first kind having ω for a period.

II. If a linear differential equation has the above form it always possesses at least one integral which is a periodic function of the third kind. The total number of linearly independent integrals of this kind is always

$$= \lambda_1 + \lambda_2 + \ldots \lambda_m,$$

where λ, ε and the fundamental equation $\Delta = 0$ have the meanings assigned to them above.

III. There exists a fundamental system of integrals consisting of

$$\lambda_1 + \lambda_2 + \ldots + \lambda_m$$

periodic functions of the third kind and in addition $n - (\lambda_1 + \lambda_2 \ldots + \lambda_m)$ functions, which are each of the form of [a polynomial in x whose coefficients are periodic functions of the second kind, all having the same multiplier] $\times\, e^{\frac{-\lambda x^2}{2\omega}}$ *i. e.* a polynomial in x whose coefficients are periodic functions of the third kind.

IV. The different multipliers of the periodic functions of the second kind are roots of the fundamental equation.

V. In order that the differential equation may have n linearly independent integrals which are periodic functions of the third kind, it is necessary and sufficient that each root of $\Delta = 0$ annul all the minors of Δ up to the order equal to the degree of multiplicity of the root.